Module 3: Problem Solving

Module Introduction

This module focuses on the concept of problem solving.

NCTM Problem-Solving Standard

Instructional programs from prekindergarten through grade 12 should enable all students to:

- build new mathematical knowledge through problem solving,
- solve problems that arise in mathematics and in other contexts,
- apply and adapt a variety of appropriate strategies to solve problems, and
- monitor and reflect on the process of mathematical problem solving.

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Before you begin this module, read the following article on Problem Solving (Reprinted with permission from Navigating Through Problem Solving and Reasoning in Grade 3, copyright 2004 by the National Council of Teachers of Mathematics. All rights reserved.) to give you some background information.

Want to Know Questions

After reading the article, write five questions you would like to be able to answer at the end of this module.

Understanding Problem Solving

As you begin this exploration of problem solving, you will need to recognize some accepted understandings that researchers use to characterize problem solving in today’s mathematics classrooms.

“Problem solving is not a specific topic to be taught but a process that permeates all mathematics” (NCTM, 2000, p.182).

“Problem solving is the cornerstone of school mathematics. Without the ability to solve problems, the usefulness and power of mathematical ideas, knowledge, and skills are severely limited” (NCTM, 2000, p.182).

“Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving
problems is not only a goal of learning mathematics but also a major means of doing so” (NCTM, 2000, p.52).

“Research informs us that young children begin school having already learned quite a lot of mathematics from their everyday problem solving, that they continue to create and acquire new mathematical procedures on their own throughout the years they spend in school, and that schools often suppress these naturally developed methods. Teachers should capitalize on the knowledge their students bring to school by having it serve as a basis for the development of mathematical knowledge that is more deeply understood and applicable to a wider variety of situations than those they have experienced in their narrowly defined worlds” (Lester, 2002, p. 77).

Teachers need to create a problem-solving atmosphere in their classrooms where children persist in their problem solving and “view mathematical problems as personal challenges and become upset if someone tells them the answer. They believe that mathematics should make sense, and they achieve personal satisfaction when they figure something out for themselves. Furthermore, they feel free to discuss their mathematical understandings both in their small groups and in the whole-class discussions, and they accept that their solutions should be explainable and justifiable” (Cobb et al., 2002, p. 72).

To become good mathematical problem solvers students must analyze the meaning of problems, decide on the most optimum strategies, select from known procedures or adapt them for the particular situation, carry out the procedures and check how effective they are, and insure the reasonableness of answers. Definitions and computational skills alone are not sufficient. (Charles & Silver, 1988; Schoenfeld, 1985). "These skills do not develop automatically; students must have opportunities to develop and practice them” (Hiebert, 2003, p. 13).

**Problem Solving in the Classroom**

Young students are by their very nature curious, and it is a teacher’s responsibility to harness that curiosity and help students to develop problem-solving skills that will equip them for the thinking involved in higher-level mathematics courses.

**Selecting Appropriate Problems**

The development of problem solving skills begins with the selection of appropriate problems. It was long thought that story problems were the way to get students thinking about mathematics processes, but it is more appropriate to select problems where the focus is on the process of solving the problem, rather than arriving at the correct answer.

"Good problems and problem-solving tasks encourage reflections and communication and can emerge from the student’s environment or from purely mathematical contexts. They generally serve multiple purposes, such as challenging students to develop and
apply strategies, introducing them to new concepts, and providing a context for using skills. They should lead somewhere, mathematically" (NCTM, 2000, p.183).

**Problem Solving Strategies**

The goal of teachers should be not only to have students produce correct answers, but also to develop their problem solving skills. It is important for teachers to model problem solving strategies by thinking aloud. Students need to feel that the classroom is a safe environment to take time to think through solutions and communicate their ideas, whether right or wrong. By modeling and encouraging discussion of ideas, teachers are helping to build on their students’ inquisitive nature. Polya (1985) provides a four-step problem-solving model that is useful for teachers to help guide their students in deepening their problem solving abilities.

1. **Understand the problem.** Find the known(s) and the unknown(s); sketch a chart picture or diagram; find the conditions that connect the unknown(s) and known(s). Do you know a definition, formula, equation, model, algorithm, rule, law, or theorem that can help you?

2. **Make a plan.** Use the following:
   - Is this problem familiar? If so, use the method you know.
   - Look for a pattern. Use inductive reasoning.
   - Guess, test, and revise; i.e., use trial and error.
   - Solve a similar, simpler problem.
   - Work backwards from the known(s) to the unknown(s).
   - Eliminate impossible situations.
   - Use direct or indirect reasoning.
   - Break the problem into parts or cases.
   - Check all possibilities.
   - Look for a “catch” or “trick” if an answer looks too obvious or impossible.
   - Restate the problem.

3. **Work the plan.** Check each step. Is it correct? Can you prove it?

4. **Look back and check.**
   - Does the result make sense?
   - Check it by reading it back into the statement of the problem. Can the result be found in a different (perhaps shorter) way?

Instruction in a general method of problem solving appears to enhance student success, especially as children reach the upper elementary grades (Hembree, 1992). Young children naturally use the first 3 steps from Polya listed above as they problem solve. Teachers can guide students' mathematical thinking about step 4 with classroom activities that include sharing solution strategies and thoughtful responses to open-ended questions.
The following describes how children approach story problems organized around the four steps.

Step 1: Understand the Problem

Young children may ask to have the problem repeated in its entirety. Or they may repeat valuable parts of information aloud: “I had 3 cats….How many are still here?” When a student does not understand some information, the teacher can help the child find a better interpretation. For example, a second grade class is working on the problem: You have two friends visit. Your father gives you 7 cupcakes. How can you share your cupcakes so that you and your friends have the same? One child draws 4 people at the top of a page. The teacher notices the error. She shows the child a small box of dolls. “Which doll are you? Which ones are your 2 friends? How many people get the cupcakes?” The child erases one of the people. The teacher gives each student 7 paper cupcakes, and the class approaches the next steps in problem solving in a variety of ways.

Step 2: Devise a Plan

Children employ a number of strategies without any formal instruction. In one sense they act impulsively and do not follow “steps” (Carpenter et al., 1990; Carpenter & Moser, 1984). Most early elementary students use informal strategies such as direct model, counting-on, or derived facts. Learning to draw simple diagrams helps students gain mathematical power. Children employ other strategies, such as guess and check or trial and error, when faced with a new situation, just as adults do. Trial and error, or trial and success approaches, are part of any problem solving environment. Teachers who encourage risk taking and promote a safe, supportive environment will not need to explicitly teach this method as a strategy.

Step 3: Carry Out the Plan

In this step, young students carry out the strategy or strategies they chose until the problem is solved. Many teachers ask children to write about their strategy. At times the children are so busy implementing a strategy that they forget where they are in the process.

When carrying out the plan, some children get discouraged and want to quit for a time. It they have spent a reasonable amount of time on a problem, they may put it aside. Perhaps they could review the problem with others, or ask how they got started. A fresh start may bring new insight into challenging problems.

Usually a problem is stated in words, which are translated into the language of math symbols. When children write a story about how they thought about the problem, they see this task as an additional challenge. They may forget what they did first, second, or third, or they may not be able to draw their solution. Teachers who think carefully about these student problems plan ahead to help students with these hurdles. They may fold the paper
into threes or provide predrawn circles. These steps take patience, flexibility, and creativity so that all children succeed.

Step 4: Look Back

Good teachers take the time to let children share their results. Children are curious and inventive. Teachers should listen to each solution. The class may need instruction in how to listen to others. Some children are too eager to share and may shout out, while others are so shy they may hold back important information. These children, along with those who have trouble expressing themselves in English, need extra attention.

When a wrong answer is given, children will change their minds when their classmates explain their answers. Teachers need to say, “Did anyone else get that answer? Who thought of it in a different way?” Teachers need to practice talking about mathematics and writing about mathematics with young students. These are essential activities in the early childhood classroom. Children may know what they want to say but have trouble getting the words out. They need time to think aloud or require support in their efforts.

Open the [PowerPoint on Problem Solving](#) to review important problem-solving concepts before you read about teaching strategies to assist students with problem solving.

**Strategies to Assist with Problem Solving**

Fraivillig (2001) identified strategies that will assist students to think deeply about their problem solving strategies and mathematical concepts. Students must share their thinking and ideas with others.

**Strategies to support students’ thinking are**

- reminding students of conceptually similar problems,
- providing background knowledge,
- leading students to revisit their solutions, and
- writing symbolic representations of solutions when appropriate.

**Strategies to extend students’ thinking are**

- maintaining high standards and expectations for all students,
- encouraging students to make generalizations,
- Listing all solution methods on the board to promote reflection,
- pushing individual students to try alternative solution methods, and
- promoting the use of more efficient solution methods (adapted from Fraivillig, 2001, p. 454 -59).
Strategies to elicit students’ thinking are

- eliciting many solution methods for one problem;
- waiting for, and listening to, students’ descriptions of solution methods;
- encouraging students to elaborate and discuss;
- using students’ explanations as a basis for the lesson’s content;
- conveying an attitude of acceptance toward students’ errors and efforts; and
- promoting collaborative problem solving.

Assessing Problem Solving

When problem solving is valued in the mathematics classroom, the assessment tasks should mirror that purpose. Assessing the problem-solving skills involved to solve mathematical tasks is difficult and more time consuming than traditional assessments. “However, it is imperative that teachers gather evidence in a variety of ways, such as through students’ work and conversations, and use that information to plan how to help individual students in a whole-class context” (NCTM, 2000, p.120).

Reflection Questions

Be prepared to discuss these questions.

1. How does problem solving connect to various other mathematical topics?
2. How will using different problem-solving strategies enhance student thinking and understanding?
3. How would you describe the importance of problem solving in the mathematics classroom to a teacher or parent who prefers the traditional focus on computation?
4. How should lessons be structured to include opportunities to develop problem solving? Give examples.

Suggested Further Readings


State Mathematics Standards: TEKS on Problem Solving

(K.13) and (1.11) and (2.12) Underlying processes and mathematical tools. The student applies Kindergarten, first grade, and second grade mathematics to solve problems connected to everyday experiences and activities in and outside of school.

The student is expected to: (A) identify mathematics in everyday situations; (B) solve problems with guidance that incorporates the processes of understanding the problem, making a plan, carrying out the plan, and evaluating the solution for reasonableness; (C) select or develop an appropriate problem-solving strategy including drawing a picture, looking for a pattern, systematic guessing and checking, or acting it out in order to solve a problem; and (D) use tools such as real objects, manipulatives, and technology to solve problems.

3.14) and (4.14) Underlying processes and mathematical tools. The student applies Grade 3 and Grade 4 mathematics to solve problems connected to everyday experiences and activities in and outside of school.

The student is expected to: (A) identify the mathematics in everyday situations; (B) solve problems that incorporate understanding the problem, making a plan, carrying out the plan, and evaluating the solution for reasonableness; (C) select or develop an appropriate problem-solving plan or strategy, including drawing a picture, looking for a pattern, systematic guessing and checking, acting it out, making a table, working a simpler problem, or working backwards to solve a problem; and (D) use tools such as real objects, manipulatives, and technology to solve problems.
Module 3: Problem Solving

Lesson 1. Problem Solving and Place Value

Introduction

This lesson was designed to focus on problem solving within the context of place value.

Objectives

At the completion of this module the participant will be able to

- determine patterns in our number system,
- determine how numbers are related to each other,
- use a Hundreds Chart to examine patterns,
- solve problems with two- and three-digit numbers, and
- select appropriate tasks and manipulatives to teach problem solving in the context of place value.

TEKS Standards Addressed

(2.1) Number, operation, and quantitative reasoning. The student understands how place value is used to represent whole numbers. The student is expected to: (A) use concrete models of hundreds, tens, and ones to represent a given whole number (up to 999) in various ways; (B) use place value to read, write, and describe the value of whole numbers to 999; and (C) use place value to compare and order whole numbers to 999 and record the comparisons using numbers and symbols (<, =, >).

(2.3) Number, operation, and quantitative reasoning. The student adds and subtracts whole numbers to solve problems. The student is expected to: (A) recall and apply basic addition and subtraction facts (to 18);

(B) model addition and subtraction of two-digit numbers with objects, pictures, words, and numbers;

2.5) Patterns, relationships, and algebraic thinking. The student uses patterns in numbers and operations. The student is expected to:

(A) find patterns in numbers such as in a 100s chart;

(3.3) Number, operation, and quantitative reasoning. The student adds and subtracts to solve meaningful problems involving whole numbers. The student is expected to:

(A) model addition and subtraction using pictures, words, and numbers;
NCTM Standards

In prekindergarten through grade 2, all students should-

- count with understanding and recognize "how many" in sets of objects;
- use multiple models to develop initial understandings of place value and the base-ten number system;
- develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections; and
- develop and use strategies for whole-number computations, with a focus on addition and subtraction.

In grades 3-4 all students should-

- understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals;
- recognize equivalent representations for the same number and generate them by decomposing and composing numbers;
- develop fluency in adding, subtracting, multiplying, and dividing whole numbers; and
- develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results.

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Materials/Handouts

1. Problem Solving Article
2. Find the Number Activity
3. Student Activity Summary Chart

Suggested Readings and Activities

- Read the material on place value and problem solving in an elementary mathematics methods text by an author such as Cathcart, Hatfield, or Van de Walle.
- Do the Find the Number Activity. Lesson 3:1. Activity 1. Turn in your activity sheet with your solutions. Use the 100’s chart at this link.
- Complete the Student Activity Summary Chart for the Find the Number Activity: Student Activity Summary Chart
- Click on the following website: http://standards.nctm.org/document/eexamples/chap4/4.5/index.htm
  1. Do the task given with the interactive hundreds board and calculator. Then click on the Patterns to 100 and Beyond link and complete the task given and answer the questions.
2. Bring your answers to the questions to the next class.
   
   Go to http://www.learner.org/resources/series32.html and register at the site (Annenberg Foundation) for free access to the materials.

1. Click on video #4 Place Value Centers (From Teaching Math: A Video Library K-4. Used with permission from Annenberg Media).
   
2. View the video and answer the analysis questions.

   a. Why did Ms. Vigstrom have students working at centers? What is the value?

   b. How was measurement used to explore place value?

   c. What is the benefit of using different manipulatives to explore the same concept?

   View the Marilyn Burns video clips linked below on addition and subtraction with Base 10 blocks and answer the discussion questions:

   http://realvideo.coe.tamu.edu:8080/ramgen/distance/mmcapraro/BaseTenBlocks2.rm

   http://realvideo.coe.tamu.edu:8080/ramgen/distance/mmcapraro/BaseTenBlocks3.rm

   1. What did you notice about the students' explanations of their actions?
   2. Did the blocks help the students to understand regrouping?

Follow-up Activity

   Make Three Flats. Answer the following questions:

7. What benefit is this activity to students in developing addition and subtraction computational understanding and proficiency?

8. Will the blocks help the students connect subtraction and addition?

Three additional activities are available:

   • Billy and the Pencils—an activity involving money
   • Fill the Cube—an activity involving estimation and subtraction
- **Billy and the Shopping Spree—an activity involving money and multiplication**

Please note this lesson is not intended for a one-hour session. Depending on the elements selected, the time frame will
Module 3: Problem Solving

Lesson 2. Problem Solving and Fractions

Introduction

This lesson was created to focus on problem solving within the context of fractions.

Objectives

At the completion of this module the participant will be able to

- place fractions within the context of the TEKS and national standards,
- determine how fractions are related to each other,
- use pattern blocks to find equivalent fractions, and
- develop a repertoire of pedagogies to problem solve with fractions.

Standards Addressed

TEKS: (K.3) Number, operation, and quantitative reasoning. The student recognizes that there are quantities less than a whole.

The student is expected to: (A) share a whole by separating it into two equal parts; and (B) explain why a given part is half of the whole.

(1.2) Number, operation, and quantitative reasoning. The student uses pairs of whole numbers to describe fractional parts of whole objects or sets of objects. The student is expected to: (A) separate a whole into two, three, or four equal parts and use appropriate language to describe the parts such as three out of four equal parts; and (B) use appropriate language to describe part of a set such as three out of the eight crayons are red.

(2.2) Number, operation, and quantitative reasoning. The student describes how fractions are used to name parts of whole objects or sets of objects. The student is expected to: (A) use concrete models to represent and name fractional parts of a whole object (with denominators of 12 or less); (B) use concrete models to represent and name fractional parts of a set of objects (with denominators of 12 or less); and (C) use concrete models to determine if a fractional part of a whole is closer to 0, 1/2, or 1.

(3.2) Number, operation, and quantitative reasoning. The student uses fraction names and symbols (with denominators of 12 or less) to describe fractional parts of whole objects or sets of objects.
The student is expected to: (A) construct concrete models of fractions; (B) compare fractional parts of whole objects or sets of objects in a problem situation using concrete models; (C) use fraction names and symbols to describe fractional parts of whole objects or sets of objects; and (D) construct concrete models of equivalent fractions for fractional parts of whole objects.

(4.2) Number, operation, and quantitative reasoning. The student describes and compares fractional parts of whole objects or sets of objects. The student is expected to: (A) use concrete objects and pictorial models to generate equivalent fractions; (B) model fraction quantities greater than one using concrete objects and pictorial models; (C) compare and order fractions using concrete objects and pictorial models; and (D) relate decimals to fractions that name tenths and hundredths using concrete objects and pictorial models.

**Curriculum Focal Points (2006)**

Grade 3:

**Number and Operations: Developing an understanding of fractions and fraction Equivalence**

Students develop an understanding of the meanings and uses of fractions to represent parts of a whole, parts of a set, or points or distances on a number line. They understand that the size of a fractional part is relative to the size of the whole, and they use fractions to represent numbers that are equal to, less than, or greater than 1. They solve problems that involve comparing and ordering fractions by using models, benchmark fractions, or common numerators or denominators. They understand and use models, including the number line, to identify equivalent fractions.

Grade 4

**Number and Operations: Developing an understanding of decimals, including the connections between fractions and decimals**

Students understand decimal notation as an extension of the base-ten system of writing whole numbers that is useful for representing more numbers, including numbers between 0 and 1, between 1 and 2, and so on. Students relate their understanding of fractions to reading and writing decimals that are greater than or less than 1, identifying equivalent decimals, comparing and ordering decimals, and estimating decimal or fractional amounts in problem solving. They connect equivalent fractions and decimals by comparing models to symbols and locating equivalent symbols on the number line.

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Readings and Activities

- Read in a methods text book the chapter on fractions.
- Complete the activity, "Three Bean Salad." Turn in your activity sheet at the next class session.
- Play “Fraction Track”. Take a screen shot when you have all your markers on the right side. Answer the “Take Time to Reflect” questions on back of the screen shot in bullet format. Bring the screen shot and answers to class.
- Go to this website http://www.learner.org and complete the following:
  1. Set up a user name and password at this site (Annenberg Foundation) and have free access to the materials.
  2. Go to Teaching Math: A Video Library, K-4 and select video #33, Fraction Strips, (From Teaching Math: A Video Library K-4. Used with permission from Annenberg Media.) View the video of Fraction Strips in which first- and second-graders make fraction pieces from paper strips and play a game that involves covering a whole strip with fractional pieces. As they play they informally add fractions and make connections from objects and actions to symbols. NCTM standards addressed: fractions and decimals, number sense and numeration, reasoning, communication.
  3. After viewing the video clip, complete a one-page reflection by addressing the following points:
     - content/process strands and TEKS
     - short summary of the lesson
     - reaction to the lesson
     - response to analysis questions
     - application in your own classroom
  4. Be prepared to discuss in class the analysis questions from the video plus the following questions:
  5. How were fraction strips used to explore fractional concepts?
  6. What is the benefit of using different manipulatives to explore the same concept?

View the Marilyn Burns video clip on Pattern Blocks. Be prepared to discuss these questions:

1. What did you notice about the students' explanations of their actions?
2. Did the pattern blocks help the students to understand equivalent fractions?

Follow-up Activity

Complete: Module 3: Lesson 2. Activity 2

Be prepared to discuss these questions:

1. What benefit is this activity to students in developing fractional equivalency understanding?
2. Explain the importance of a unit whole in this activity.
3. Is 1/2 always equal to 1/2?

Extension to the lesson

Choose one of the following five articles on fractional problem solving from the NCTM journal *Teaching Children Mathematics*. Write a short summary and a short reaction to each article. Then describe the fractional problem solving that the authors discussed in each article. What TEKS does each article address?

- (2000, Nov) *Oil and water Don’t Mix But They Do Teach Fractions*, 174-178.
Checking for Understanding - Module 3: Lesson 2.

1. __________ should be an integral part of computation development to keep students’ attention on the meanings of the operations and the expected size of the results.
2. It is important to give students ample opportunity to develop __________ before talking about common denominators and other rules of computation.
3. The meaning of fraction computation should be connected to __________ computation.
4. Students will develop strategies to solve problems with fractions, and their __________ will contribute to the development of more standard methods.
5. Using their own invented strategies, students will see that many correct solutions are found for addition and subtraction without ever getting a __________.
6. An understanding of the algorithms for addition and subtraction is heavily dependent on a conceptual understanding of __________.
7. A separate algorithm for __________ in addition and subtraction is not necessary; include them in all the activities with addition and subtraction.
8. Having students change mixed numbers to improper fractions for multiplication is not necessary; they can do the multiplication by using __________.
9. When shifting to traditional multiplication algorithm from contextual problems, __________ can be used as a model.
10. When common-denominator algorithm is used for division, the problem becomes one of dividing __________.
11. The common-denominator algorithm relies on the __________ concept of division.

This concludes the Problem Solving Module. To continue to the Student Thinking Module, click on the right arrow button below.