Introduction

*Principles and Standards for School Mathematics* (NCTM 2000) states that “problem solving is the cornerstone of school mathematics. Without the ability to solve problems, the usefulness and power of mathematical ideas, knowledge, and skills are severely limited” (p. 182). Mathematical investigations that challenge students to deal with nonroutine problems and situations should be a regular part of Standards-based instruction at all levels.

Just as solving problems can help students make sense of their changing world, so also justifying solutions and communicating the results of mathematical investigations can help elementary school students develop and expand their reasoning abilities. One goal is for students to develop ways of thinking about mathematics that encourage sense making and reasoning about solutions and strategies. The mathematics classroom is the main environment in which students speak and write mathematics. Hence, it is essential that teachers offer students opportunities to communicate mathematically by having them make, test, discuss, and refine conjectures, ultimately accepting or rejecting them.

The investigations in this book engage students in extended tasks that enable them to look for relationships among concepts in the five content strands—number and operations, algebra, geometry, measurement, and data analysis and probability. Each investigation enables students to focus on one strand in depth. At the same time, the activities illustrate how a carefully chosen mathematical task can bridge content areas. For example, although the focus of the algebra investigation is on looking for patterns, students work with the factors of a number (requiring knowledge from the number and operations strand) and build geometric representations for the number (using a skill from the geometry strand).
Aspects of Problem Solving

Good problems challenge students to develop and apply strategies, serve as a means to introduce new concepts, and offer a context for using skills. Problem solving is not a specific topic to be taught but a process that permeates all mathematics. What behaviors might a visitor expect to observe in a classroom that makes problem solving a focus? According to Principles and Standards for School Mathematics, students should—

build new mathematical knowledge through problem solving;
solve problems that arise in mathematics and in other contexts;
apply and adapt a variety of appropriate strategies to solve problems; and
monitor and reflect the process of mathematical problem solving. (NCTM 2000, p. 402)

Students build new mathematical knowledge through problem solving

Students can learn new mathematical concepts and skills through problem solving. A successful problem-centered approach uses interesting problems to motivate students to expend time and energy and be persistent in seeking solutions. Under the guidance of a teacher who encourages students to reason creatively and make connections between ideas, students can discover new mathematical concepts, techniques, and relationships. New ideas often emerge from discussions among students; teachers should guide such discussions carefully so that the students learn the difference between correct mathematical reasoning and incorrect reasoning and between sound problem-solving strategies and unsound ones. The teacher must summarize classroom discussions so that the students are aware of new knowledge and skills that they have derived from the problem-solving experience.

The investigation And We All Go Marching affords students the opportunity to observe how a change in one variable affects a second variable. Students' reasoning about composition and decomposition becomes more sophisticated as they explore the effects of rearranging pairs of shapes in Cut It Apart, Put It Together. In Grant Avenue Elementary School Reading Certificates, students use data in a way not often seen in grade 3—to make, apply, and evaluate rules for awarding reading certificates.

Students solve problems that arise in mathematics and in other contexts

The investigations in this book pose problems in contexts that appeal to third graders. For instance, students construct pedometers in Walking into Place Value and they work with them to explore the relationships among the ones, tens, and hundreds places. In the activity And We All Go Marching, students explore different ways to arrange students in parade formations. In Grant Avenue Elementary School Reading Certificates, students reason and solve problems as they design a system for assigning points to students who read books of varying difficulty and length. In each of these investigations, students grapple with problems that arise in real-world situations. The problems are interesting and challenging vehicles for exploring mathematics and thinking about relationships.

Students apply and adapt a variety of strategies to solve problems

As students explore problems, they need to consider a variety of strategies to investigate the situation. In And We All Go Marching, students explore the parade-formation task with concrete materials, represent the solutions on a grid, and display data in a table and a graph. The varied representations afford students many different ways to explore a problem and enable those with different learning styles to benefit from the problem-solving experience.

Students monitor and reflect on the process of mathematical problem solving

As students work through good mathematical tasks, they reflect on their work to determine what strategies are effective and where they need to make adjustments. In Cut It Apart, Put It Together, students investigate the effects of cutting shapes and rearranging the pieces to make different shapes. Students are encouraged to make a prediction first and then test that prediction. By observing what happens when they make the initial cuts, students become much more proficient in making future predictions. How Many Are Too Many requires that students monitor their results as they place different numbers of nonstandard units into a container (a "boat") in an attempt to sink it in a tub of water. Through a trial-and-error process, students are able to determine the minimum number of units needed to sink the container. Reflection is an important aspect of this problem-solving process.

Aspects of Reasoning

Reasoning develops over time as teachers facilitate discussions of rich tasks and help students learn "to construct valid arguments and to evaluate the arguments of others" (NCTM 2000, p. 188). As students reason about mathematics, they should—

make and investigate mathematical conjectures;
develop and evaluate mathematical arguments and proofs; and
select and use various types of reasoning and methods of proof. (NCTM 2000, p. 402)
Students make and investigate mathematical conjectures

Students need to learn that making conjectures on the basis of patterns is a natural part of mathematical thinking. In Walking into Place Value, students make and investigate conjectures about what happens to the values of the digits in the ones and tens places when the last digit in a number is a 9 and a 1 is added to the number. In Cat It Apart, Put It Together, students predict what will happen when a figure is cut by a line segment, and then test their prediction by using an applet or cutting the figure.

Students develop and evaluate mathematical arguments and proofs

Students use shapes to validate the arguments that they make in Cat It Apart, Put It Together. In Grant Avenue Elementary School Reading Certificates, students develop a set of rules for awarding reading certificates and then present their rules to the rest of the class. By applying their rules to the reading profiles of other children, the students can work together to evaluate the effectiveness and fairness of their own rules and those of others. In And We All Go Marching, students write a summary that explains the relationships that they discovered among different parade formations.

Students select and use various types of reasoning and methods of justification

Throughout the investigations, students have an opportunity to justify their reasoning in several ways. The data-analysis investigation, Grant Avenue Elementary School Reading Certificates, encourages students to reason by applying their proposed plans to new data. The geometry tasks in Cat It Apart, Put It Together stimulate students to reason by making predictions and then testing those predictions by using an applet or by actually cutting the figures. In the measurement investigation How Many Are Too Many? students use a trial-and-error process as they refine their judgments about the number of objects needed to make the container sink.

The Role of the Teacher

As students explore the tasks in this book, their teachers should monitor their activities and foster the interactions necessary to maintain high levels of reasoning. (See Stein and Smith [1998].) The tasks are mathematically rich, but if teachers provide too many clues or too much specific help early in the process, they can still the deep thinking that the tasks can elicit from students. The challenge for teachers is to facilitate students' communication about a task without directing the students toward a particular solution.

From detailed observations in an elementary school classroom using a Standards-based curriculum, Fraivillig (2001) has identified various strategies that are essential to helping students think deeply about mathematical ideas and share their thinking with others. These approaches fall into three broad categories: eliciting students' thinking, supporting students' thinking, and extending students' thinking. Descriptions of the strategies in each category are summarized in figure 1.

### Strategies to elicit students' thinking
- Elicit many solution methods for one problem.
- Wait for, and listen to, students' descriptions of solution methods.
- Encourage students to elaborate and discuss.
- Solicit students' explanations as a basis for the lesson's content.
- Convey an attitude of acceptance toward students' errors and efforts.
- Promote collaborative problem solving.

### Strategies to support students' thinking
- Remind students of conceptually similar problems.
- Provide background knowledge.
- Lead students through "instant replays" (Revisit student solutions.)
- Write symbolic representations of solutions when appropriate.

### Strategies to extend students thinking
- Maintain high standards and expectations for all students.
- Encourage students to make generalizations.
- List all solution methods on the board to promote reflection.
- Push individual students to try alternative solution methods.
- Promote the use of more efficient solution methods.

Adapted from Fraivillig (2001, pp. 454–59)

![Fig. 1. Strategies to advance students' thinking](image)

In addition, research from the QUASAR project in urban schools with underachieving students found that the following actions by teachers were associated with higher performance by their students on a test of problem solving (Henningsen and Stein 1997; Stein, Grover, and Henningsen 1996, Smith and Stein 1998):

- Teachers press for explanations and meaning.
- Teachers have capable students model high-level performance.
- Teachers allow appropriate time for students to explore the task, to think, and to make sense of mathematics for themselves.
- Teachers note conceptual connections.
- Teachers build on students' prior knowledge.

Teachers who engage in behaviors like those identified by Fraivillig and by the QUASAR researchers help students develop their reasoning and problem-solving abilities. Teachers can use the following questions to help elicit students' reasoning:

- "Why?"
- "How do you know?"

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“What other problems can you remember that are similar to this one?”

“What other ways could you solve this problem?”

“Do you agree with this approach to this problem? Why, or why not?”

Such questions can stimulate important teacher-student discourse that will strengthen the reasoning abilities of all students and can engage students in mathematical communication in the classroom.

The role of the teacher is indispensable, and the investigations in this book are designed to help teachers encourage problem solving and reasoning in elementary school mathematics students. Engaging students in these processes is an essential component of developing their mathematical power.